Jerzy Kotas

THE AXIOMATIZATION OF S. JAŚKOWSKI’S DISCUSSIVE SYSTEM

S. Jaśkowski in [2] and [3] has determined through interpretation a new logical system $D_2$, very interesting in many respects, which he has called a discussive system. L. Dubikajtis and N. C. A. da Costa in [1] gave an infinite axiom set for that system. The present paper aims at demonstrating that $D_2$ is a finitely axiomatizable. We shall make use of the Łukasiewicz bracketless notation. The symbols $K$, $A$, $C$, $N$, $M$ and $L$ will denote conjunction, disjunction, material implication, negation, possibility, and necessity, respectively. Logical systems will be treated as the sets of formulae.

In determining the system $D_2$ Jaśkowski employs the modal system $S_5$ of Lewis. He enriches the set of logical connectives of the system $S_5$ with three additional connectives $K_d$, $C_d$, and $E_d$ which he calls discussive conjunction, discussive implication, and discussive equivalence, respectively. These logical connectives he defines as follows:

**Definition 1.** $K_d pq = df KpMq$;

**Definition 2.** $C_d pq = df CMpq$;

**Definition 3.** $E_d pq = df KdCdpqCdqp$.

The system $D_2$ is the least set of formulae $\alpha$ fulfilling the following conditions:

1. There are present in $\alpha$ only the signs of propositional variables and the signs $K_d$, $C_d$, $E_d$, $A$ and $N$;

2. The expression $M\alpha_L$, where $\alpha_L$ is the expression obtained from $\alpha$ by eliminating the symbols $K_d$, $C_d$, $E_d$, according to their definitions.
Let the symbols $LS_5$ and $M - S_5$ denote, respectively, the set of all those theses of the system $S_5$ at the beginning of which there is the symbol $L$, and the set of all formulae of the system $S_5$ which after being preceded by the sign $M$ become theses of the system $S_5$. Since we treat the logical systems as sets of formulae, $LS_5$ and $M - S_5$ are certain logical systems. Making use of $C$, $N$ and $L$ it is possible to define in a well known way all remaining logical connectives of the system $S_5$; thus considering the systems $S_5$, $LS_5$ and $M - S_5$ we can confine ourselves to the set of formulae where, apart from variables, there are only the signs $C$, $N$ and $L$. For the same reason we may put that $K_d$, $C_d$, $A$, $N$ are the only signs of the logical connectives occurring in the formulae of the system $D_2$.

Let $A$ be the set consisting of the following formulae:

\[(A_1) \quad LCpCNpq,\]
\[(A_2) \quad LCCpqCCqrCpr,\]
\[(A_3) \quad LCCNppp,\]
\[(A_4) \quad LCNp,\]
\[(A_5) \quad LCNCpqCLpLq,\]
\[(A_6) \quad LCNpLNLp.\]

In further considerations we shall employ the following rules of deduction:

\[(R_1) : \text{substitution rule;}\]
\[(R_2) : \text{if } L\alpha, \text{ and } LC\alpha\beta, \text{ then } L\beta;\]
\[(R_3) : \text{if } \alpha, \text{ then } L\alpha;\]
\[(R_4) : \text{if } L\alpha, \text{ then } \alpha;\]
\[(R_5) : \text{if } N LN\alpha, \text{ then } \alpha.\]

The symbol $Cn(A; R_1, \ldots, R_n)$ denote the set of all formulae which are the consequences of the set $A$ with respect to the rules of deduction $(R_1), \ldots, (R_n)$.

**Lemma 1.** $Cn(A; R_1, R_2, R_3) = LS_5$.

**Lemma 2.** $Cn(A; R_1, R_2, R_3, R_4) = S_5$.

**Lemma 3.** $Cn(A; R_1, R_2, R_3, R_4, R_5) = M - S_5$. 
Since from $C\alpha\beta \in M - S5$ and $C\beta\alpha \in M - S5$ it does not follow that $CN\alpha N\beta \in M - S5$, then it does not follow that $CN\alpha N\beta \in M - S5$ the implication as meant in [4]. For the same reasons $C_d$ is not an implication as meant above, in the system $D_2$. It is easy to prove that the strict implication, denoted here by the symbol $C_S$, as well as the logical connective $I$ which is defined in the following way:

**Definition 4.** $Ipq \equiv_d NK_dANrRNANpq$ are implications in the system $M - S5$ and $D_2$, respectively, in the sense as determined in [4].

The interpretation $i_1$ of the system $D_2$ in the system $M - S5$, and the interpretation $i_2$ of the system $M - S5$ in the system $D_2$ we determine in the following way:

I. For any formulae $\alpha$ and $\beta$ of the system $D_2$:  
   i) $i_1(\alpha) = \alpha$, when $\alpha$ is a propositional variable,  
   ii) $i_1(N\alpha) = Ni_1(\alpha)i_1(\beta)$,  
   iii) $i_1(A\alpha\beta) = CNi_1(\alpha)i_1(\beta)$,  
   iv) $i_1(K_d\alpha\beta) = Nan_i_1(\alpha)LNi_1(\beta)$,  
   v) $i_1(C_d\alpha\beta) = CNLN_i_1(\alpha)i_1(\beta)$;  

II. For any formulae $\alpha$ and $\beta$ of the system $M - S5$:  
   i) $i_2(\alpha) = \alpha$ when $\alpha$ is a propositional variable,  
   ii) $i_2(N\alpha) = Ni_2(\alpha)$,  
   iii) $i_2(C\alpha\beta) = A\alpha_i_2(\alpha)i_2(\beta)$,  
   iv) $i_2(L\alpha) = NK_dANpN\alpha_i_2(\alpha)$.  

**Lemma 4.** $\gamma Cs i_1(Ipq)CSpq \cap \gamma C_S CSpqi_1(Ipq) \cap \gamma \in M - S5$.  

**Lemma 5.** $\gamma Ii_2(CSpq)Ipq \cap \gamma IIPqi_2(CSpq) \cap \gamma \in D_2$.  

From the Lemmas 4 and 5 it follows that the interpretation $i_1$ turns the implication $I$ in the strict implication $C_S$, and the interpretation $i_2$ turns the implication $C_S$ in $I$.

**Lemma 6.** The interpretations $i_1$ and $i_2$ establish the equivalence of the systems $D_2$ and $M - S5$.  

Because $D_2$ and $M - S5$ are equivalent systems, from the notes at the end of [4], from Theorem 4 put in [5], p. 361, and from Lemmas 3 and 6 it follows
Theorem. $D_2$ is a finitely axiomatizable system.

In the above mentioned Theorem 4 of [5] a method is given which effectively enables to obtain the axioms of a logical system if the axioms of the equivalent logical system are known. Making use of that method we obtain the following

Corollary. The formulae $i_2(A_i)$, $i = 1, \ldots, 6$, $i_2i_1(Fpq)Fpq$, $IFpq_i_1(Fpq)$, where instead of the symbol $F$, symbols $K_d$, $C_d$, $A$ should be put in turn, and the rules $(R_i, i_2)$, $i = 1, 2, 3, 4, 5$, connected through the interpretation $i_2$ with the rules $(R_i)$, $i = 1, 2, 3, 4, 5$, constitute the complete axiom set of the system $D_2$.

References


Institute of Mathematics
Nicholas Copernicus University
Toruń